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On the steady separated flow around an inclined flat plate

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An inviscid analytic model is proposed for the steady separated flow around an inclined flat plate. With the plate normal to the stream, the model reduces to the wake-source model of Parkinson & Jandali originally developed for flow external to a symmetrical two-dimensional bluff body and its wake. At any other inclination, the Kutta condition is satisfied at both leading and trailing edges of the plate, and, in the limit that the angle of attack approaches zero, classical airfoil theory is recovered. A boundary condition is formulated based on some experimental results of Abernathy, but no additional empirical information is required. The predicted pressure distributions on the wetted surface for a wide range of angle attack are found to be in good agreement with experimental data, especially at smaller angles of attack. An extension to include a leading-edge separation bubble is explored and results are satisfactory.

1. Introduction

To the aerodynamicist, the importance of studying the flow around an inclined flat plate can be perceived through two limiting situations. At small angles of inclination, the flat plate act as a thin airfoil to produce lift, fundamental to the theory of aerodynamics. When the plate is normal to the stream, it becomes a bluff body which is also of aeronautical interest (e.g. a spoiler on an airfoil) when a large value of drag force is required.

In a uniform stream approaching a flat plate at a small angle of inclination, α, the Kutta condition applied at the trailing edge ensures finite velocity locally and allows the determination of circulation and lift. In this classical airfoil theory, the velocity around the leading edge is infinite because the flow has to turn around a sharp edge, resulting in infinite suction. *Visualized Flow* (Japan Society of Engineers 1988, figures 127 and 128) shows that a separation bubble is located at the leading edge for $3° \leqslant \alpha \leqslant 7°$, making the pressure finite as supported by the measurements of McCullough & Gault (1951).

The free-streamline theory introduced by Kirchhoff (1869) which made use of the method of conformal mapping developed by Helmholtz (1868) treated the flow normal to a flat plate. Their model requires that the pressure along the separation streamlines bounding the wake is equal to its free-stream value. Roshko (1954) modified the free-streamline theory such that the pressure over the initial portions of the free streamlines is equal to that in the wake measured experimentally. The predicted pressure distribution is in good agreement in comparison with experimental data by Fage & Johansen (1927, hereinafter referred to as FJ) Further extensions of the free-streamline theory allowing a continuous variation of flow in the whole range ($0 < \alpha \leqslant 90°$) were carried out by Mimura (1958), Wu (1962) and Abernathy (1962). Good agreement is found between theoretical and experimental pressure distributions when $30° \leqslant \alpha \leqslant 90°$ but discrepancy becomes obvious when $\alpha \leqslant 15°$, as reported by Mimura. In the models for fully developed wake flows and partially developed wake flows by Wu (1962), it has been shown that the latter model can be reduced to the classical airfoil theory. A thorough review of the free-streamline theories can be found in Wu (1972).

A wake-source model developed by Parkinson & Jandali (1970, hereinafter referred to as PJ) for flow external to a symmetrical bluff body and its wake has been shown not only to give predictions in good agreement with experimental data from FJ and Roshko's theory in the case of the flow normal to a flat plate but also in cases such as a circular cylinder, 90°-wedge and an elliptical cylinder. The model makes use of conformal mapping and mathematical singularities in a different and simpler way from the above-mentioned free-streamline theories. Replacing conformal mapping by vortex-lattice discretization, Bearman & Fackrell (1975) extended the model to axisymmetric bluff bodies, such as a circular disk and a sphere. Kiya & Arie (1977) modified the wake-source model by including the far-wake displacement effect which was originally put forward by Woods (1961).

This paper presents a model for steady separated flow around an inclined flat plate. With the plate normal to the stream, the model reduces to the wake-source model of PJ. At any other inclination, the Kutta condition is satisfied at both leading and trailing edges of the plate. Circulation is added to the flow so that as the angle of attack approaches zero, classical airfoil theory is recovered, as in the model for partially developed wake flows developed by Wu (1962). An attempt of this kind has been reported by Bearman & Fackrell but similar to the free-streamline theories, there is deteriorating agreement between the experimental results and the theoretical and numerical results as α becomes 15° or less. The suitability of the zero-total-vorticity condition around bodies generating substantial lift and the availability of an additional empirical parameter, such as the net vorticity, were discussed in their paper. Better agreement is found if the pressure at the centre of the plate is specified.

The model for complete flow separation described herein shows good agreement with experimental data for $14.85° \leqslant \alpha \leqslant 90°$ by introducing a new boundary condition suitably developed from the physical evidence based on some experimental data from Abernathy, although no additional empirical parameters are required. Extensions of this model to include incomplete flow separation or a separation bubble at the leading edge when $\alpha = 5.85°$ are included and results are satisfactory.

2. Theory

2.1. *Flow model for completely separated flow*

Complete separation is assumed behind the plate such that the pressure on the plate exposed to the wake is constant. The flat plate of unit length with uniform stream U approaching at incidence α in the physical plane, as shown in figure 1, is conformally mapped to a circle of unit radius in the transform plane by the Joukowsky transformation,

$$z = \frac{1}{4}\left(\mathrm{e}^{\mathrm{i}\alpha}\zeta + \frac{1}{\mathrm{e}^{\mathrm{i}\alpha}\zeta}\right) + \frac{1}{2}.$$

Two surface sources of strengths $2Q_1$ and $2Q_2$ and at angular locations δ_1 and δ_2 are added to the circumference of the circle to create stagnation streamlines at two points

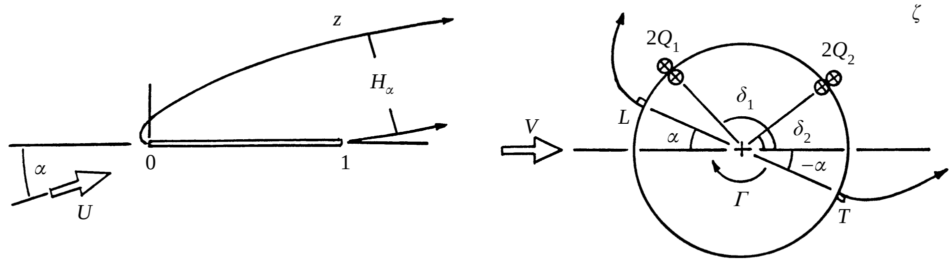

FIGURE 1. Physical and transform planes.

(180° apart on the circle) which coincide with the tips of the plate in the physical plane and are also the critical points in the mapping. Unlike the original wake-source model, the plate is at an arbitrary incidence so that symmetry can no longer be applied to determine the strengths and the locations of these sources. A vortex of strength Γ is added to the centre of the circle so that the complex potential is

$$F(\zeta) = V\left\{\zeta+\frac{1}{\zeta}+\mathrm{i}\gamma\ln\zeta+q_1\ln(\zeta-\mathrm{e}^{\mathrm{i}\delta_1})+q_2\ln(\zeta-\mathrm{e}^{\mathrm{i}\delta_2})-\tfrac{1}{2}(q_1+q_2)\ln\zeta\right\},$$

where
$$\gamma = \frac{\Gamma}{2\pi V}, \quad q_1 = \frac{Q_1}{\pi V}, \quad q_2 = \frac{Q_2}{\pi V}.$$

The velocity V in the ζ-plane is related to the velocity U in the z-plane by

$$U\mathrm{e}^{-\mathrm{i}\alpha} = \frac{V}{(\mathrm{d}z/\mathrm{d}\zeta)|_\infty} \Rightarrow U = 4V.$$

The unknowns include Q_1, δ_1, Q_2, δ_2 and Γ, and can be determined by the following appropriate boundary conditions:

(*a*) Flow stagnates at the leading and trailing edges in the ζ-plane, where $\zeta_L = \mathrm{e}^{\mathrm{i}(\pi-\alpha)}$, $\zeta_T = \mathrm{e}^{-\mathrm{i}\alpha}$

$$w(\zeta)|_{L,T} = \left.\frac{\mathrm{d}F}{\mathrm{d}\zeta}\right|_{L,T} = 0, \tag{1}$$

so that two stagnation streamlines are formed at the critical points. This can be considered as a form of Kutta condition because the velocity is finite at these points in the z-plane.

(*b*) By using Bernoulli's equation, the pressure at the leading and trailing edges in the z-plane is set equal to the pressure in the wake

$$C_p|_{L,T} = 1-\frac{1}{U^2}\left|\frac{\mathrm{d}F/\mathrm{d}\zeta}{\mathrm{d}z/\mathrm{d}\zeta}\right|^2_{L,T} = C_{pb}, \tag{2}$$

where C_{pb} is the base pressure coefficient.

In total, four boundary conditions result from conditions (1) and (2). If $\alpha = 90°$, by symmetry

$$Q_1 = Q_2, \quad \delta_1 = -\delta_2, \quad \Gamma = 0.$$

Therefore, the original wake-source model of PJ is recovered. These four conditions, even if $\alpha \neq 90°$, are sufficient to solve the problem when the vortex is excluded. Some preliminary calculations, however, show that, with $\Gamma = 0$, the predicted pressure distributions are in good agreement with experiment only if $\alpha \geqslant 50°$. This is similar to the results reported in Bearman & Fackrell. (Note that their definition of α is different from the notation used here.) Apparently, the complex potential is incomplete without the vortex and its strength is to be determined by an extra boundary condition.

Before discussing this new boundary condition, the significance of the vortex, which contributes to the circulation around the flat plate, is to be examined. In the limit that $\alpha \to 0°$, the wake behind the plate will diminish and so will the strengths of the sources. The Kutta condition at the trailing edge leads to the determination of the strength of the vortex, which is

$$w(\zeta_T) = V\left(1 - \frac{1}{\zeta_T^2} + \mathrm{i}\frac{\gamma}{\zeta_T}\right) = 0, \quad Q_1 = Q_2 = 0,$$

$$\frac{\Gamma}{2\pi V} = 2\sin\alpha \Rightarrow \Gamma = \pi U \sin\alpha.$$

Therefore, classical airfoil theory is recovered. As can be seen, the inclusion of the vortex not only improves the agreement between theoretical and experimental results, but also allows the Kutta condition to be properly satisfied.

2.2. *Deduction of fifth boundary condition*

According to the experimental results reported by Abernathy,

$$D \propto c \sin\alpha,$$

where D is the 'separation between free-vortex layers', c is the length of the plate and $40° \leqslant \alpha \leqslant 90°$. Here, only the data with least blockage are considered because the present model is for unconfined flow. In the context of this steady flow model, the following empirical relation is deduced

$$H_\alpha \propto c \sin^m \alpha,$$

where H_α is the asymptotic downstream spacing of the separation streamlines at α, and $m = 1$. Note that H_α is perpendicular to the upstream velocity, U. (The necessity of exponent m will be obvious later.) This provides a condition to link the total source strength at any α to that at $\alpha = 90°$, i.e.

$$\frac{Q_1 + Q_2}{2Q} = \frac{H_\alpha}{H_{90°}} = \left(\frac{\sin\alpha}{\sin 90°}\right)^m, \tag{3}$$

where Q is the source strength at $\alpha = 90°$. The first equality is derived from the equation of continuity, since

$$2Q = UH_{90°}.$$

With $m = 1$, good agreement is found between the prediction and the measured pressure distribution from FJ on the upstream side of the plate at $\alpha = 69.85°$, $49.85°$, $29.85°$. The case for $\alpha = 90°$ can be found in PJ. Note that Abernathy's data are valid for $40° \leqslant \alpha \leqslant 90°$ since the chamfer angle at the leading and trailing edges of the plate used is 25°. The value of m, however, must be increased to 1.5 for $\alpha = 14.85°$ and $m = 1.90$ for $\alpha = 5.85°$ to achieve satisfactory agreement. This undesirable variation of

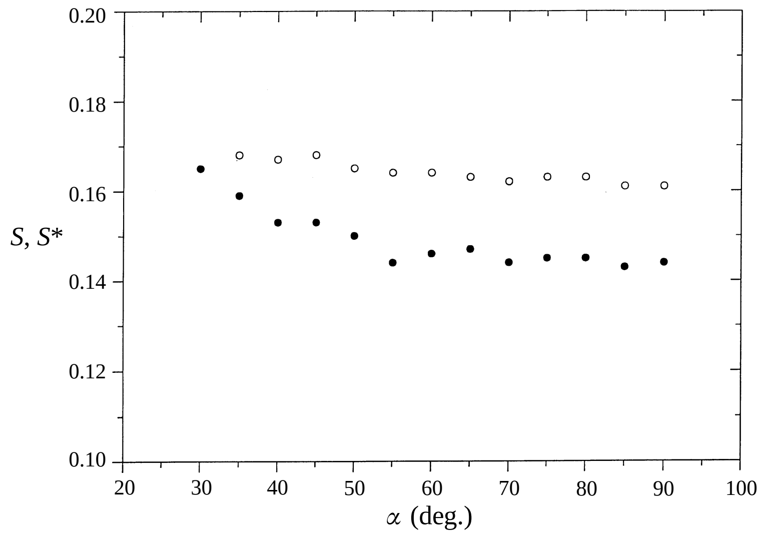

FIGURE 2. Variations of Strouhal numbers ($K = 13.98$) from Abernathy (1962). ●, S^*; ○, S.

m may be explained by the fact that D is no longer well defined as α gets smaller, owing to possible flow reattachment on the upper surface of the plate, but it makes the model unsatisfactory. Another model is now considered.

As originally proposed by Roshko and adopted by Abernathy, a modified Strouhal number S^*, defined as

$$S^* = \frac{nD}{U_s} = \frac{n\,1.41\,c\sin\alpha}{(1-C_p^*)^{1/2}\,U_0},$$

is found to be a constant in Abernathy's data. The frequency of vortex shedding is n, while U_s is the maximum mean speed in the free-vortex layers from the inclined plate, C_p^* is the pressure coefficient behind the plate (which is the same as C_{pb} used here) and U_0 is the free-stream speed.

If one proposed yet another modified Strouhal number S^{**} based on D^* as

$$S^{**} = \frac{nD^*}{U_s},$$

where $D^* = (1-C_p^*)^{1/2} c\sin\alpha$, then

$$S^{**} = \frac{n(1-C_p^*)^{1/2}\,c\sin\alpha}{(1-C_p^*)^{1/2}\,U_0} = \frac{nc\sin\alpha}{U_0}.$$

In other words, $S^{**} = S$, the conventional Strouhal number based on projected plate width. Figure 2 is the comparison of S^* and S in Abernathy's data for the case of minimum blockage. Indeed, S is also independent of a wide range of α, especially at smaller angles of attack. Therefore, it may be more suitable to use the condition

$$\frac{Q_1+Q_2}{2Q} = \frac{H_\alpha}{H_{90^\circ}} = \frac{(1-C_{pb_\alpha})^{1/2}\sin\alpha}{(1-C_{pb_{90^\circ}})^{1/2}\sin 90^\circ}, \tag{4}$$

instead of (3) as the new boundary condition.

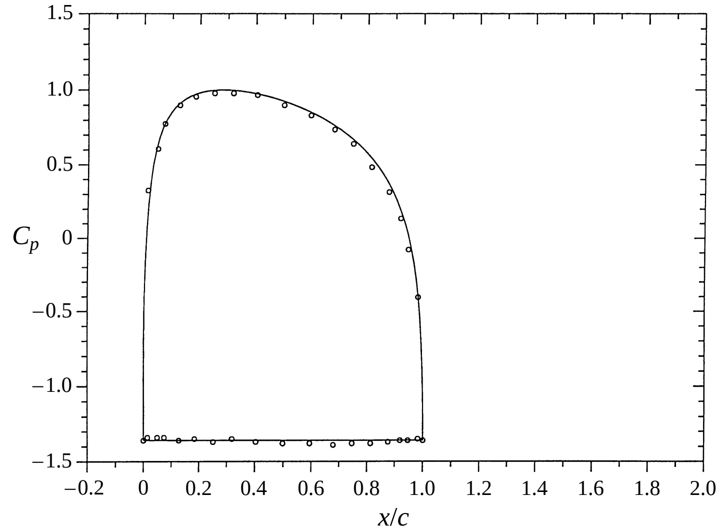

FIGURE 3. Pressure distribution ($\alpha = 69.85°$, $C_{pb} = -1.36$). ○, Fage & Johansen; ——, present work.

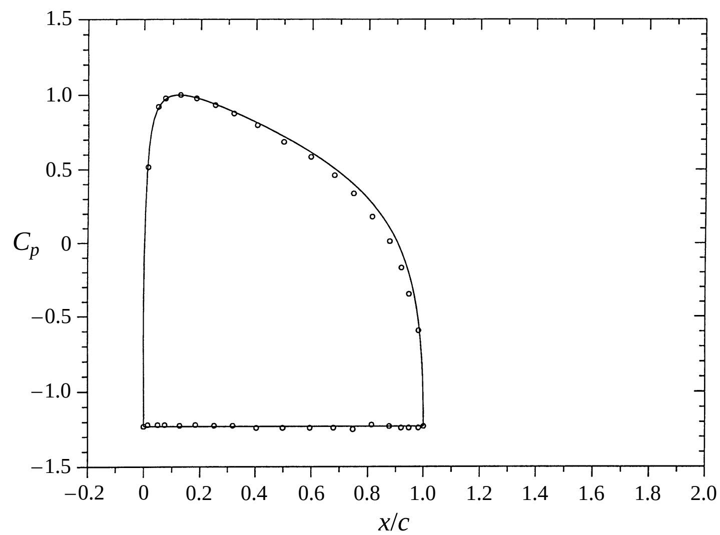

FIGURE 4. Pressure distribution ($\alpha = 49.85°$, $C_{pb} = -1.23$). ○, Fage & Johansen; ——, present work.

2.3. *Results and discussion*

By adopting condition (4) as the fifth boundary condition, the predicted pressure distributions for $\alpha = 69.85°$, $49.85°$, $29.85°$, $14.85°$ are shown in figures 3–6 in comparison with the data from FJ. The agreement is quite satisfactory, especially without introducing any extra parameter, such as m which varies with α in (3). Equation (4), which requires neither any specification of net vorticity in the model *a priori*, nor additional empirical information other than the base pressure, is the preferred boundary condition in this model.

The variation of base pressure near the trailing edge as shown in figure 6 signifies

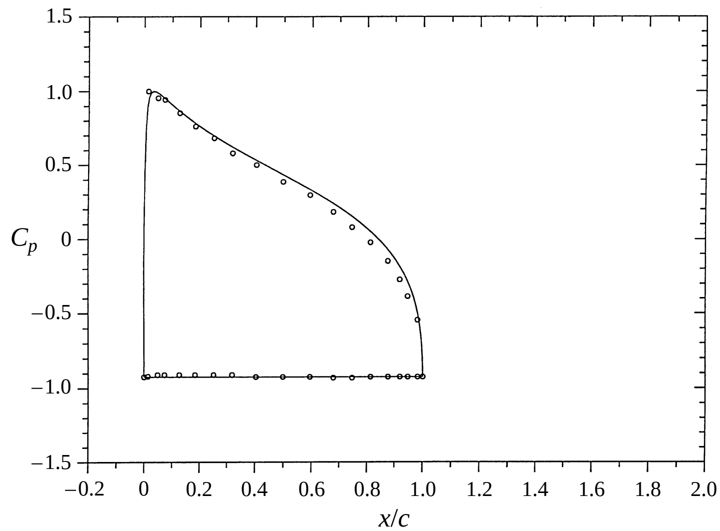

FIGURE 5. Pressure distribution ($\alpha = 29.85°$, $C_{pb} = -0.924$). ○, Fage & Johansen; ——, present work.

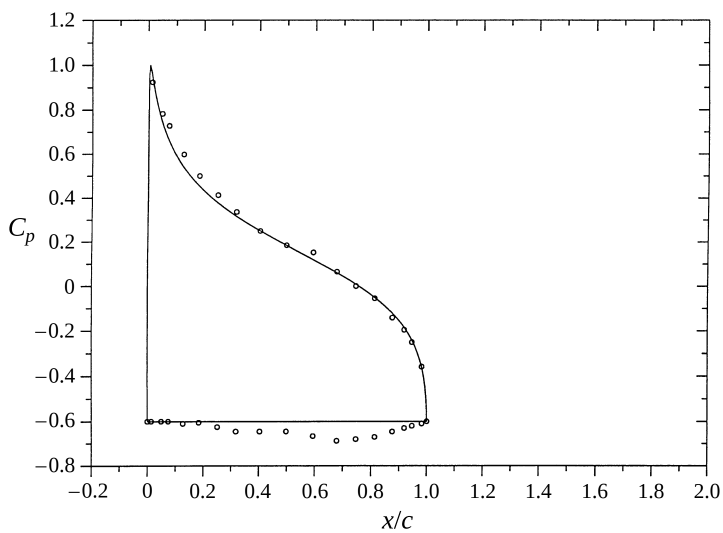

FIGURE 6. Pressure distribution ($\alpha = 14.85°$, $C_{pb} = -0.60$). ○, Fage & Johansen; ——, present work.

that a different flow pattern may take place if α is further reduced. Wu (1962) described this type of flow as partially developed wake flows in which the near-wake constant pressure covers only the initial portion of the suction side of the plate in contrast to fully developed wake flows.

2.4. *Flow model for incompletely separated flow*

Because of the change in flow pattern over the upper surface of the plate, a model is proposed such that the pressure over an initial portion of the streamline emanating from the leading edge is constant. A model can be formulated by introducing more surface sources to the complex potential described in §2.1.

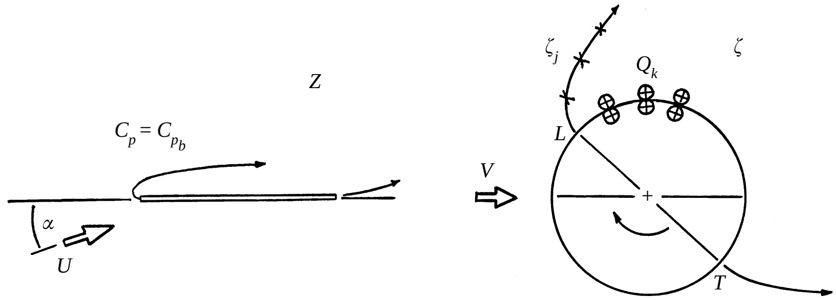

FIGURE 7. Physical and transform planes.

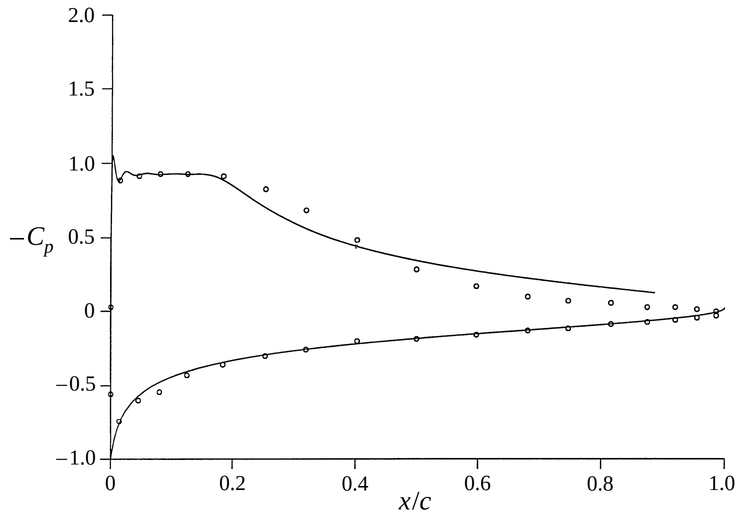

FIGURE 8. Pressure distribution ($\alpha = 5.85°$, $C_{pb} = -0.928$). ○, Fage & Johansen; ——, present work.

While keeping the same conformal mapping as in §2.1, the complex potential for flow around a circle of unit radius, plus N_q surface sources of strengths $2Q_k$ located at angular positions δ_k, and a vortex at the origin, as shown in figure 7, is given by

$$F(\zeta) = V\left\{\zeta + \frac{1}{\zeta} + \mathrm{i}\gamma \ln \zeta + \sum_{k=1}^{N_q} q_k [\ln(\zeta - \mathrm{e}^{\mathrm{i}\delta_k}) - \tfrac{1}{2}\ln\zeta]\right\},$$

where
$$\gamma = \frac{\Gamma}{2\pi V}, \quad q_k = \frac{Q_k}{\pi V}.$$

The Kutta condition is satisfied at the leading and trailing edges, i.e. (i) $w(\zeta)|_{L,T} = 0$. At $\zeta_j = r_j \mathrm{e}^{\mathrm{i}\theta_j}$, where $j = 1, 2, \ldots, N_r$, (ii) $C_p = C_{pb}$, and (iii) $\mathrm{Im}(F(\zeta)) = \mathrm{Im}(F(\zeta))|_L$.

Condition (ii) is the specification of the base pressure on N_r discrete points in the flow field, and (iii) is to ensure that these points lie along the separation streamline from the leading edge. Therefore, the unknowns are q_k, δ_k (where $k = 1, 2, \ldots, N_q$), γ and r_j (where $j = 1, 2, \ldots, N_r$). Note that the pressure at the leading and trailing edges is not specified but replaced by condition (ii). In addition, condition (3) or (4) is no longer appropriate in this model simply because the wake is not fully developed and flow reattachment might be involved.

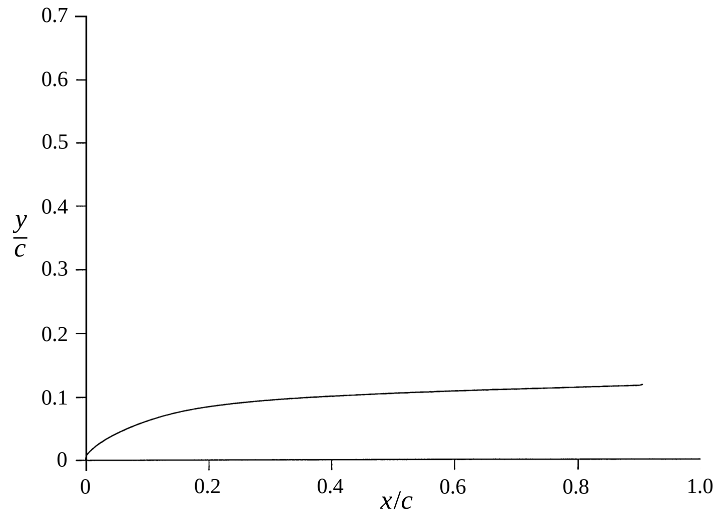

FIGURE 9. Separation streamline from leading edge ($\alpha = 5.85°$).

2.5. *Results and discussion*

With five surface sources ($N_q = 5$), nine discrete points ($N_r = 9$) and the specification of

$$\theta_{2j+1} = \delta_j,$$
$$\theta_{2j} = \tfrac{1}{2}(\delta_{j+1} + \delta_{j+2}),$$

where $j = 0, 1, \ldots, 4$, the pressure distribution from the model is compared with the data from FJ at $\alpha = 5.85°$ in figure 8, with the shape of the leading-edge streamline depicted in figure 9. The agreement on the lower surface is quite satisfactory. On the upper surface, discrete points on the initial portion of the separation streamline have $C_p = C_{pb} = -0.928$ as specified, although there is a small variation of pressure $(-0.87 \geqslant C_p \geqslant -1.1)$ near the leading edge. The agreement downstream of this portion of constant pressure becomes less satisfactory, perhaps because the effects of boundary layer become more pronounced.

3. Discussion and concluding remarks

An analytical model for flow around a flat plate at an arbitrary angle of inclination with complete separation is proposed. The paper considers the problem again because previous models fail to reproduce satisfactory results at low angles of attack α, owing to the neglect of circulation Γ. The key to including this additional unknown in the problem was to derive the new boundary condition from the well-documented experimental data without requiring any additional empirical input. (The justification of assuming constant Strouhal number at $\alpha = 14.85°$, $29.85°$ lies on the good agreement between the experimental data and the prediction from the model.) This is worthwhile without considering extensions to other body shapes. However, it would be relatively simple to apply the theory to other shapes which experience the same leading-edge separation as the flat plate, e.g. circular-arc plates, Kármán–Trefftz airfoils with sharp leading edge, both classes which could be conformally mapped from the ζ-plane of the paper. The predicted pressure distributions, in general, agree well with experimental data. It has been shown that the model provides a smooth transition between classical airfoil theory as $\alpha \to 0°$ and the wake-source model of PJ as $\alpha \to 90°$. The inclusion of the vortex to generate circulation around the plate perhaps explains why this vortex is also required in flow models for lifting bodies as reported in Jandali & Parkinson (1970), Parkinson & Yeung (1987) and Yeung & Parkinson (1993). Although it is interesting to extend the application of (4) to bodies of other shapes, meaningful conclusions cannot be drawn until supporting data for other shapes are available. The contribution of this paper is to show that with an appropriate boundary condition derived from physical measurements, a potential flow model can still work well for separated flow. Moreover, it has been demonstrated that experimental data, besides being used for validating theoretical predictions, can be useful for the construction of conditions for theoretical formulation.

For the model dealing with incomplete separation, the separation streamline emanating from the leading edge with constant pressure over an initial portion is created such that the flow turns around a leading edge with finite curvature. Therefore, the unrealistic infinite suction peak resulting from flow turning around the sharp leading edge, as predicted by classical airfoil theory, can be avoided. The specification of pressure on points along the wake boundary does not introduce any additional empiricism because the pressure on the initial portion of the free streamline is the same as the base pressure, as in Roshko (1954), and the locations of these points are the solution of the equations. Because the predicted streamline does not join with the plate after separation or the other streamline from the trailing edge, the model may be suitable for describing the flow outside the boundary layer in the case of turbulent reattachment as given in the picture of flow visualization from Werlé (1974).

From *Visualized Flow* (Japan Society of Mechanical Engineers 1988) at $3° \leqslant \alpha \leqslant 7°$ and Werlé (1974) at $\alpha = 2.5°$, a bubble of recirculating fluid is found at the leading edge of a flat plate. No attempt has been made to model this type of flow in all the above-mentioned free-streamline theories and wake-source models. Although results are still far from being final, some preliminary work has been carried out to create a closed separation streamline to model a separation bubble at the leading edge of the plate at small incidence. At $\alpha = 5.85°$, by specifying the location of reattachment and satisfying the condition of pressure gradient at reattachment, the pressure distributions on the wetted surface of the plate and the portion of the plate downstream of reattachment are in good agreement with the data by FJ. The thickness of the closed streamline predicted by the model is in the same order of magnitude as the data from McCullough & Gault, although the data are for a thin double-wedge airfoil at $\alpha = 6°$. Nevertheless, the predicted pressure on the separation streamline is very different from the experimental data. Perhaps the specification of pressure on the closed streamline, such as in the model for incomplete separation, will improve the agreement.

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